10.7 Circles in the Coordinate Plane

Texas Essential Knowledge and Skills

G.12.E

Essential Question

What is the equation of a circle with center (h, k) and radius r in the coordinate plane?

EXPLORATION 1 The Equation of a Circle with Center at the Origin

Work with a partner. Use dynamic geometry software to construct and determine the equations of circles centered at (0, 0) in the coordinate plane, as described below.

a. Complete the first two rows of the table for circles with the given radii. Complete the other rows for circles with radii of your choice.

b. Write an equation of a circle with center (0, 0) and radius r.

Radius	Equation of circle
1	
2	

EXPLORATION 2 The Equation of a Circle with Center (h, k)

Work with a partner. Use dynamic geometry software to construct and determine the equations of circles of radius 2 in the coordinate plane, as described below.

a. Complete the first two rows of the table for circles with the given centers. Complete the other rows for circles with centers of your choice.

b. Write an equation of a circle with center (h, k) and radius 2.

c. Write an equation of a circle with center (h, k) and radius r.

Center	Equation of circle
(0, 0)	
(2, 0)	

EXPLORATION 3 Deriving the Standard Equation of a Circle

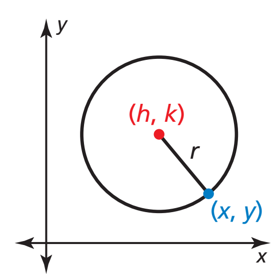
USING PROBLEM-SOLVING STRATEGIES

To be proficient in math, you need to explain correspondences between equations and graphs.

Work with a partner. Consider a circle with radius r and center (h, k).

Write the Distance Formula to represent the distance d between a point (x, y) on the circle and the center (h, k) of the circle. Then square each side of the Distance Formula equation.

How does your result compare with the equation you wrote in part (c) of Exploration 2?



Communicate Your Answer

4. What is the equation of a circle with center (h, k) and radius r in the coordinate plane?

5. Write an equation of the circle with center $(4, -1)$ and radius 3.

10.7 Lesson

Core Vocabulary

standard equation of a circle, *p. 580*

Previous

completing the square

What You Will Learn

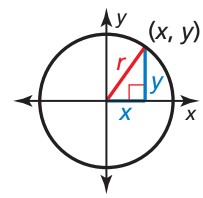
- Write and graph equations of circles.
- Write coordinate proofs involving circles.
- Solve real-life problems using graphs of circles.

Writing and Graphing Equations of Circles

Let (x, y) represent any point on a circle with center at the origin and radius r. By the Pythagorean Theorem (Theorem 9.1),

$$x^2 + y^2 = r^2.$$

This is the equation of a circle with center at the origin and radius r.



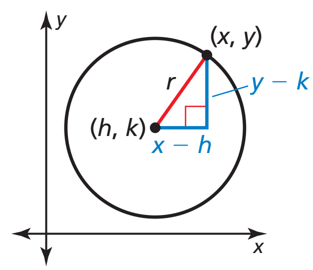
Core Concept

Standard Equation of a Circle

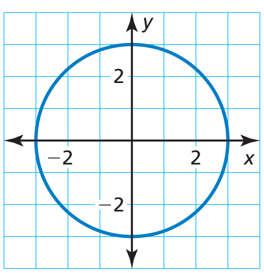
Let (x, y) represent any point on a circle with center (h, k) and radius r. By the Pythagorean Theorem (Theorem 9.1),

$$(x - h)^2 + (y - k)^2 = r^2.$$

This is the **standard equation of a circle** with center (h, k) and radius r.



EXAMPLE 1 Writing the Standard Equation of a Circle



Write the standard equation of each circle.

a. the circle shown at the left

b. a circle with center $(0, -9)$ and radius 4.2

SOLUTION

a. The radius is 3, and the center is at the origin.

$(x - h)^2 + (y - k)^2 = r^2$ Standard equation of a circle

$(x - 0)^2 + (y - 0)^2 = 3^2$ Substitute.

$x^2 + y^2 = 9$ Simplify.

▶ The standard equation of the circle is $x^2 + y^2 = 9$.

b. The radius is 4.2, and the center is at $(0, -9)$.

$(x - h)^2 + (y - k)^2 = r^2$ Standard equation of a circle

$(x - 0)^2 + [y - (-9)]^2 = 4.2^2$ Substitute.

$x^2 + (y + 9)^2 = 17.64$ Simplify.

▶ The standard equation of the circle is $x^2 + (y + 9)^2 = 17.64$.

Monitoring Progress

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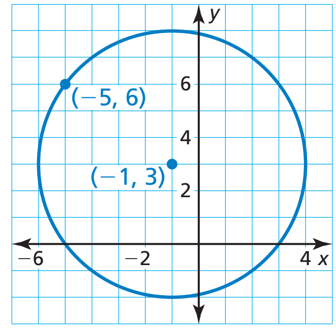
Write the standard equation of the circle with the given center and radius.

1. center: $(0, 0)$, radius: 2.5

2. center: $(-2, 5)$, radius: 7

EXAMPLE 2 Writing the Standard Equation of a Circle

The point $(-5, 6)$ is on a circle with center $(-1, 3)$. Write the standard equation of the circle.



SOLUTION

To write the standard equation, you need to know the values of h, k, and r. To find r, find the distance between the center and the point $(-5, 6)$ on the circle.

$$\begin{aligned} r &= \sqrt{[-5-(-1)]^2 + (6-3)^2} && \text{Distance Formula} \\ &= \sqrt{(-4)^2 + 3^2} && \text{Simplify.} \\ &= 5 && \text{Simplify.} \end{aligned}$$

Substitute the values for the center and the radius into the standard equation of a circle.

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 && \text{Standard equation of a circle} \\ [x-(-1)]^2 + (y-3)^2 &= 5^2 && \text{Substitute } (h, k) = (-1, 3) \text{ and } r = 5. \\ (x+1)^2 + (y-3)^2 &= 25 && \text{Simplify.} \end{aligned}$$

▶ The standard equation of the circle is $(x + 1)^2 + (y - 3)^2 = 25$.

REMEMBER

To complete the square for the expression $x^2 + bx$, add the square of half the coefficient of the term bx.

$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

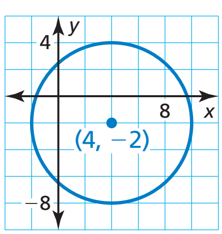
EXAMPLE 3 Graphing a Circle

The equation of a circle is $x^2 + y^2 - 8x + 4y - 16 = 0$. Find the center and the radius of the circle. Then graph the circle.

SOLUTION

You can write the equation in standard form by completing the square on the x-terms and the y-terms.

$$\begin{aligned} x^2 + y^2 - 8x + 4y - 16 &= 0 && \text{Equation of circle} \\ x^2 - 8x + y^2 + 4y &= 16 && \text{Isolate constant. Group terms.} \\ x^2 - 8x + 16 + y^2 + 4y + 4 &= 16 + 16 + 4 && \text{Complete the square twice.} \\ (x-4)^2 + (y+2)^2 &= 36 && \text{Factor left side. Simplify right side.} \\ (x-4)^2 + [y-(-2)]^2 &= 6^2 && \text{Rewrite the equation to find the center and the radius.} \end{aligned}$$



▶ The center is $(4, -2)$, and the radius is 6. Use a compass to graph the circle.

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3. The point $(3, 4)$ is on a circle with center $(1, 4)$. Write the standard equation of the circle.

4. The equation of a circle is $x^2 + y^2 - 8x + 6y + 9 = 0$. Find the center and the radius of the circle. Then graph the circle.

Writing Coordinate Proofs Involving Circles

EXAMPLE 4 Writing a Coordinate Proof Involving a Circle

Prove or disprove that the point $(\sqrt{2}, \sqrt{2})$ lies on the circle centered at the origin and containing the point (2, 0).

SOLUTION

The circle centered at the origin and containing the point (2, 0) has the following radius.

$$r = \sqrt{(x - h)^2 + (y - k)^2} = \sqrt{(2 - 0)^2 + (0 - 0)^2} = 2$$

So, a point lies on the circle if and only if the distance from that point to the origin is 2. The distance from $(\sqrt{2}, \sqrt{2})$ to (0, 0) is

$$d = \sqrt{(\sqrt{2} - 0)^2 + (\sqrt{2} - 0)^2} = 2.$$

▶ So, the point $(\sqrt{2}, \sqrt{2})$ lies on the circle centered at the origin and containing the point (2, 0).

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5. Prove or disprove that the point $(1, \sqrt{5})$ lies on the circle centered at the origin and containing the point (0, 1).

Solving Real-Life Problems

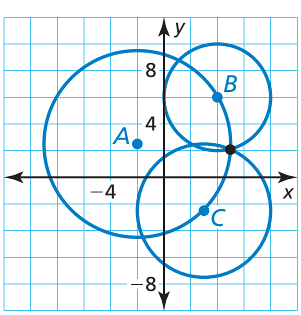
EXAMPLE 5 Using Graphs of Circles

The epicenter of an earthquake is the point on Earth's surface directly above the earthquake's origin. A seismograph can be used to determine the distance to the epicenter of an earthquake. Seismographs are needed in three different places to locate an earthquake's epicenter.

Use the seismograph readings from locations A, B, and C to find the epicenter of an earthquake.

- The epicenter is 7 miles away from $A(-2, 2.5)$.
- The epicenter is 4 miles away from $B(4, 6)$.
- The epicenter is 5 miles away from $C(3, -2.5)$.

SOLUTION



The set of all points equidistant from a given point is a circle, so the epicenter is located on each of the following circles.

$\odot A$ with center $(-2, 2.5)$ and radius 7

$\odot B$ with center $(4, 6)$ and radius 4

$\odot C$ with center $(3, -2.5)$ and radius 5

To find the epicenter, graph the circles on a coordinate plane where each unit corresponds to one mile. Find the point of intersection of the three circles.

▶ The epicenter is at about (5, 2).

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6. Why are three seismographs needed to locate an earthquake's epicenter?

10.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

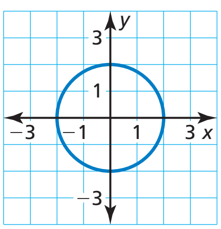
1. **VOCABULARY** What is the standard equation of a circle?

2. **WRITING** Explain why knowing the location of the center and one point on a circle is enough to graph the circle.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, write the standard equation of the circle. *(See Example 1.)*

3.



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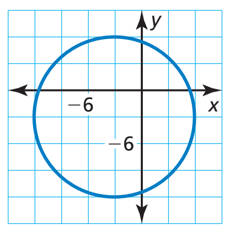


5. a circle with center (0, 0) and radius 7

6. a circle with center (4, 1) and radius 5

7. a circle with center $(-3, 4)$ and radius 1

8. a circle with center $(3, -5)$ and radius 7

In Exercises 9–11, use the given information to write the standard equation of the circle. *(See Example 2.)*

9. The center is (0, 0), and a point on the circle is (0, 6).

10. The center is (1, 2), and a point on the circle is (4, 2).

11. The center is (0, 0), and a point on the circle is $(3, -7)$.

12. **ERROR ANALYSIS** Describe and correct the error in writing the standard equation of a circle.

The standard equation of a circle with center $(-3, -5)$ and radius 3 is $(x - 3)^2 + (y - 5)^2 = 9$.

In Exercises 13–18, find the center and radius of the circle. Then graph the circle. *(See Example 3.)*

13. $x^2 + y^2 = 49$

14. $(x + 5)^2 + (y - 3)^2 = 9$

15. $x^2 + y^2 - 6x = 7$

16. $x^2 + y^2 + 4y = 32$

17. $x^2 + y^2 - 8x - 2y = -16$

18. $x^2 + y^2 + 4x + 12y = -15$

In Exercises 19–22, prove or disprove the statement. *(See Example 4.)*

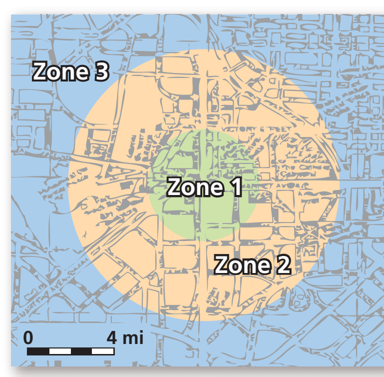
19. The point (2, 3) lies on the circle centered at the origin with radius 8.

20. The point $(4, \sqrt{5})$ lies on the circle centered at the origin with radius 3.

21. The point $(\sqrt{6}, 2)$ lies on the circle centered at the origin and containing the point $(3, -1)$.

22. The point $(\sqrt{7}, 5)$ lies on the circle centered at the origin and containing the point (5, 2).

23. **MODELING WITH MATHEMATICS** A city's commuter system has three zones. Zone 1 serves people living within 3 miles of the city's center. Zone 2 serves those between 3 and 7 miles from the center. Zone 3 serves those over 7 miles from the center. *(See Example 5.)*



   a. Graph this situation on a coordinate plane where each unit corresponds to 1 mile. Locate the city's center at the origin.

   b. Determine which zone serves people whose homes are represented by the points (3, 4), (6, 5), (1, 2), (0, 3), and (1, 6).

24. MODELING WITH MATHEMATICS Telecommunication towers can be used to transmit cellular phone calls. A graph with units measured in kilometers shows towers at points (0, 0), (0, 5), and (6, 3). These towers have a range of about 3 kilometers.

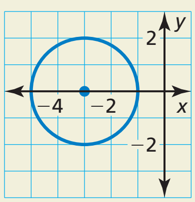
a. Sketch a graph and locate the towers. Are there any locations that may receive calls from more than one tower? Explain your reasoning.

b. The center of City A is located at (−2, 2.5), and the center of City B is located at (5, 4). Each city has a radius of 1.5 kilometers. Which city seems to have better cell phone coverage? Explain your reasoning.

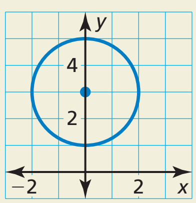
25. REASONING Sketch the graph of the circle whose equation is $x^2 + y^2 = 16$. Then sketch the graph of the circle after the translation $(x, y) \rightarrow (x - 2, y - 4)$. What is the equation of the image? Make a conjecture about the equation of the image of a circle centered at the origin after a translation m units to the left and n units down.

26. HOW DO YOU SEE IT? Match each graph with its equation.

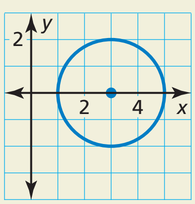
a.



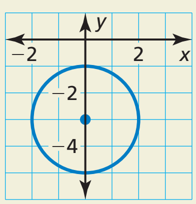
b.



c.



d.



A. $x^2 + (y + 3)^2 = 4$ **B.** $(x - 3)^2 + y^2 = 4$

C. $(x + 3)^2 + y^2 = 4$ **D.** $x^2 + (y - 3)^2 = 4$

27. USING STRUCTURE The vertices of $\triangle XYZ$ are $X(4, 5)$, $Y(4, 13)$, and $Z(8, 9)$. Find the equation of the circle circumscribed about $\triangle XYZ$. Justify your answer.

28. THOUGHT PROVOKING A circle has center (h, k) and contains point (a, b). Write the equation of the line tangent to the circle at point (a, b).

MATHEMATICAL CONNECTIONS In Exercises 29–32, use the equations to determine whether the line is *a tangent*, *a secant*, *a secant that contains the diameter*, or *none of these*. Explain your reasoning.

29. Circle: $(x - 4)^2 + (y - 3)^2 = 9$
Line: $y = 6$

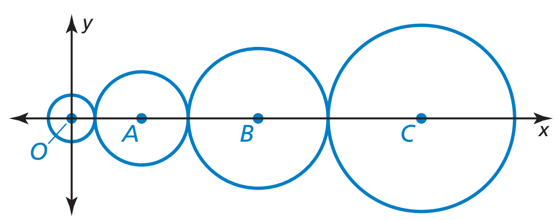
30. Circle: $(x + 2)^2 + (y - 2)^2 = 16$
Line: $y = 2x - 4$

31. Circle: $(x - 5)^2 + (y + 1)^2 = 4$
Line: $y = \frac{1}{5}x - 3$

32. Circle: $(x + 3)^2 + (y - 6)^2 = 25$
Line: $y = -\frac{4}{3}x + 2$

33. MAKING AN ARGUMENT Your friend claims that the equation of a circle passing through the points (−1, 0) and (1, 0) is $x^2 - 2yk + y^2 = 1$ with center $(0, k)$. Is your friend correct? Explain your reasoning.

34. REASONING Four tangent circles are centered on the x-axis. The radius of $\odot A$ is twice the radius of $\odot O$. The radius of $\odot B$ is three times the radius of $\odot O$. The radius of $\odot C$ is four times the radius of $\odot O$. All circles have integer radii, and the point (63, 16) is on $\odot C$. What is the equation of $\odot A$? Explain your reasoning.



Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Identify the arc as a *major arc*, *minor arc*, or *semicircle*. Then find the measure of the arc. *(Section 10.2)*

35. $\overparen{RS}$ **36.** $\overparen{PR}$

37. $\overparen{PRT}$ **38.** $\overparen{ST}$

39. $\overparen{RST}$ **40.** $\overparen{QS}$