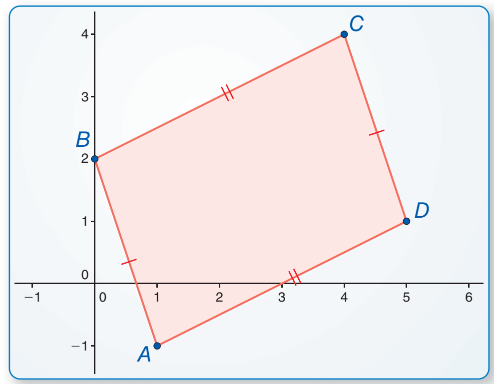
7.3 Proving That a Quadrilateral Is a Parallelogram

Texas Essential Knowledge and Skills
G.2.B
G.5.C
G.6.E

Essential Question How can you prove that a quadrilateral is a parallelogram?

EXPLORATION 1 Proving That a Quadrilateral Is a Parallelogram

Work with a partner. Use dynamic geometry software.



Sample

Points
$A(1, -1)$
$B(0, 2)$
$C(4, 4)$
$D(5, 1)$
Segments
$AB = 3.16$
$BC = 4.47$
$CD = 3.16$
$DA = 4.47$

REASONING

To be proficient in math, you need to know and flexibly use different properties of objects.

a. Construct any quadrilateral *ABCD* whose opposite sides are congruent.

b. Is the quadrilateral a parallelogram? Justify your answer.

c. Repeat parts (a) and (b) for several other quadrilaterals. Then write a conjecture based on your results.

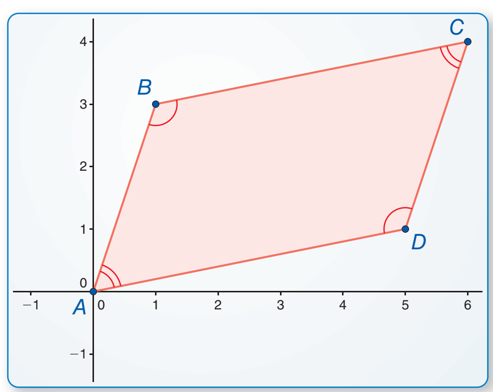
d. Write the converse of your conjecture. Is the converse true? Explain.

EXPLORATION 2 Proving That a Quadrilateral Is a Parallelogram

Work with a partner. Use dynamic geometry software.

a. Construct any quadrilateral *ABCD* whose opposite angles are congruent.

b. Is the quadrilateral a parallelogram? Justify your answer.



Sample

Points
$A(0, 0)$
$B(1, 3)$
$C(6, 4)$
$D(5, 1)$
Angles
$\angle A = 60.26°$
$\angle B = 119.74°$
$\angle C = 60.26°$
$\angle D = 119.74°$

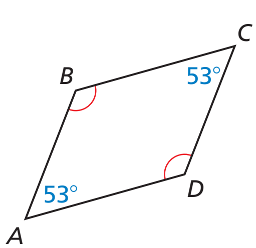
c. Repeat parts (a) and (b) for several other quadrilaterals. Then write a conjecture based on your results.

d. Write the converse of your conjecture. Is the converse true? Explain.

Communicate Your Answer

3. How can you prove that a quadrilateral is a parallelogram?

4. Is the quadrilateral at the left a parallelogram? Explain your reasoning.



7.3 Lesson

Core Vocabulary

Previous
diagonal
parallelogram

What You Will Learn

- Identify and verify parallelograms.
- Show that a quadrilateral is a parallelogram in the coordinate plane.

Identifying and Verifying Parallelograms

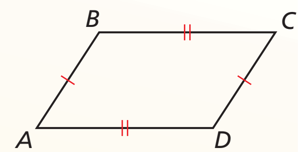
Given a parallelogram, you can use the Parallelogram Opposite Sides Theorem (Theorem 7.3) and the Parallelogram Opposite Angles Theorem (Theorem 7.4) to prove statements about the sides and angles of the parallelogram. The converses of the theorems are stated below. You can use these and other theorems in this lesson to prove that a quadrilateral with certain properties is a parallelogram.

Theorems

Theorem 7.7 Parallelogram Opposite Sides Converse

If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

If $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$, then *ABCD* is a parallelogram.

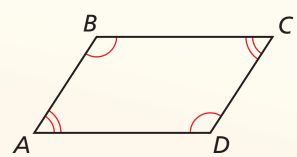


Theorem 7.8 Parallelogram Opposite Angles Converse

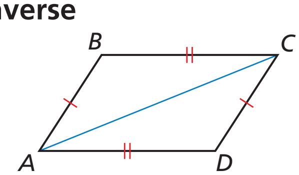
If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

If $\angle A \cong \angle C$ and $\angle B \cong \angle D$, then *ABCD* is a parallelogram.

Proof Ex. 39, p. 387



PROOF **Parallelogram Opposite Sides Converse**



Given $\overline{AB} \cong \overline{CD}$, $\overline{BC} \cong \overline{DA}$

Prove *ABCD* is a parallelogram.

Plan for Proof

a. Draw diagonal $\overline{AC}$ to form $\triangle ABC$ and $\triangle CDA$.

b. Use the SSS Congruence Theorem (Thm. 5.8) to show that $\triangle ABC \cong \triangle CDA$.

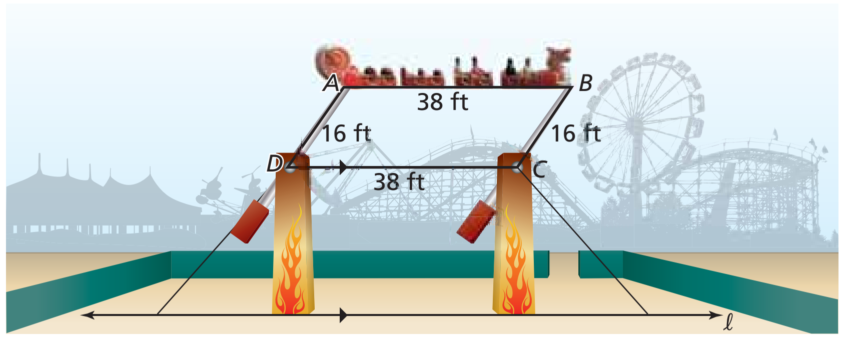
c. Use the Alternate Interior Angles Converse (Thm. 3.6) to show that opposite sides are parallel.

Plan in Action

	STATEMENTS	REASONS
a.	**1.** $\overline{AB} \cong \overline{CD}$, $\overline{BC} \cong \overline{DA}$	**1.** Given
	2. Draw $\overline{AC}$.	**2.** Through any two points, there exists exactly one line.
	3. $\overline{AC} \cong \overline{CA}$	**3.** Reflexive Property of Congruence (Thm. 2.1)
b.	**4.** $\triangle ABC \cong \triangle CDA$	**4.** SSS Congruence Theorem (Thm. 5.8)
c.	**5.** $\angle BAC \cong \angle DCA$, $\angle BCA \cong \angle DAC$	**5.** Corresponding parts of congruent triangles are congruent.
	6. $\overline{AB} \parallel \overline{CD}$, $\overline{BC} \parallel \overline{DA}$	**6.** Alternate Interior Angles Converse (Thm. 3.6)
	7. *ABCD* is a parallelogram.	**7.** Definition of parallelogram

EXAMPLE 1 Identifying a Parallelogram

An amusement park ride has a moving platform attached to four swinging arms. The platform swings back and forth, higher and higher, until it goes over the top and around in a circular motion. In the diagram below, $\overline{AD}$ and $\overline{BC}$ represent two of the swinging arms, and $\overline{DC}$ is parallel to the ground (line ℓ). Explain why the moving platform $\overline{AB}$ is always parallel to the ground.



SOLUTION

The shape of quadrilateral *ABCD* changes as the moving platform swings around, but its side lengths do not change. Both pairs of opposite sides are congruent, so *ABCD* is a parallelogram by the Parallelogram Opposite Sides Converse.

By the definition of a parallelogram, $\overline{AB} \parallel \overline{DC}$. Because $\overline{DC}$ is parallel to line ℓ, $\overline{AB}$ is also parallel to line ℓ by the Transitive Property of Parallel Lines (Theorem 3.9). So, the moving platform is parallel to the ground.

Monitoring Progress

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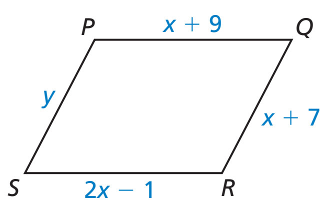
1. In quadrilateral *WXYZ*, $m\angle W = 42°$, $m\angle X = 138°$, and $m\angle Y = 42°$. Find $m\angle Z$. Is *WXYZ* a parallelogram? Explain your reasoning.

EXAMPLE 2 Finding Side Lengths of a Parallelogram



For what values of *x* and *y* is quadrilateral *PQRS* a parallelogram?

SOLUTION

By the Parallelogram Opposite Sides Converse, if both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram. Find *x* so that $\overline{PQ} \cong \overline{SR}$.

$PQ = SR$	Set the segment lengths equal.
$x + 9 = 2x - 1$	Substitute $x + 9$ for *PQ* and $2x - 1$ for *SR*.
$10 = x$	Solve for *x*.

When $x = 10$, $PQ = 10 + 9 = 19$ and $SR = 2(10) - 1 = 19$. Find *y* so that $\overline{PS} \cong \overline{QR}$.

$PS = QR$	Set the segment lengths equal.
$y = x + 7$	Substitute *y* for *PS* and $x + 7$ for *QR*.
$y = 10 + 7$	Substitute 10 for *x*.
$y = 17$	Add.

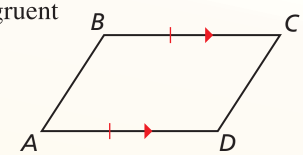
When $x = 10$ and $y = 17$, $PS = 17$ and $QR = 10 + 7 = 17$.

▶ Quadrilateral *PQRS* is a parallelogram when $x = 10$ and $y = 17$.

Theorems

Theorem 7.9 Opposite Sides Parallel and Congruent Theorem

If one pair of opposite sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.

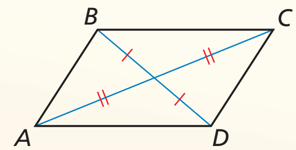


If $\overline{BC} \parallel \overline{AD}$ and $\overline{BC} \cong \overline{AD}$, then $ABCD$ is a parallelogram.

Proof Ex. 40, p. 387

Theorem 7.10 Parallelogram Diagonals Converse

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



If $\overline{BD}$ and $\overline{AC}$ bisect each other, then $ABCD$ is a parallelogram.

Proof Ex. 41, p. 387

EXAMPLE 3 Identifying a Parallelogram

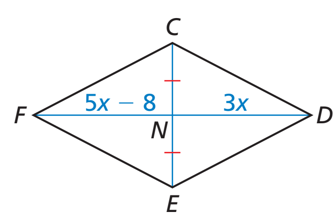
The doorway shown is part of a building in England. Over time, the building has leaned sideways. Explain how you know that $SV = TU$.

SOLUTION

In the photograph, $\overline{ST} \parallel \overline{UV}$ and $\overline{ST} \cong \overline{UV}$. By the Opposite Sides Parallel and Congruent Theorem, quadrilateral $STUV$ is a parallelogram. By the Parallelogram Opposite Sides Theorem (Theorem 7.3), you know that opposite sides of a parallelogram are congruent. So, $SV = TU$.

EXAMPLE 4 Finding Diagonal Lengths of a Parallelogram

For what value of x is quadrilateral $CDEF$ a parallelogram?



SOLUTION

By the Parallelogram Diagonals Converse, if the diagonals of $CDEF$ bisect each other, then it is a parallelogram. You are given that $\overline{CN} \cong \overline{EN}$. Find x so that $\overline{FN} \cong \overline{DN}$.

$FN = DN$	Set the segment lengths equal.
$5x - 8 = 3x$	Substitute $5x - 8$ for FN and $3x$ for DN.
$2x - 8 = 0$	Subtract $3x$ from each side.
$2x = 8$	Add 8 to each side.
$x = 4$	Divide each side by 2.

When $x = 4$, $FN = 5(4) - 8 = 12$ and $DN = 3(4) = 12$.

▶ Quadrilateral $CDEF$ is a parallelogram when $x = 4$.

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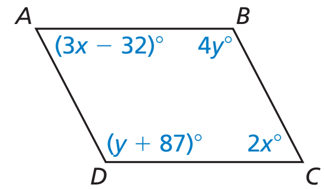
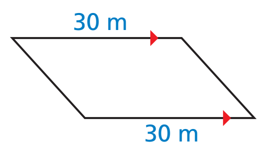
2. For what values of x and y is quadrilateral $ABCD$ a parallelogram? Explain your reasoning.



State the theorem you can use to show that the quadrilateral is a parallelogram.

3.



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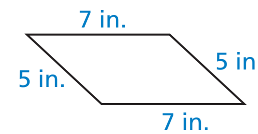

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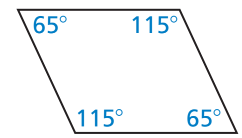
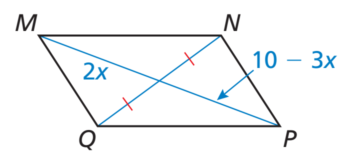
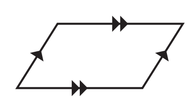
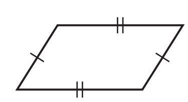
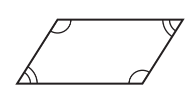
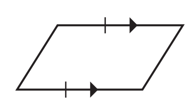
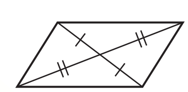


6. For what value of x is quadrilateral $MNPQ$ a parallelogram? Explain your reasoning.



Concept Summary

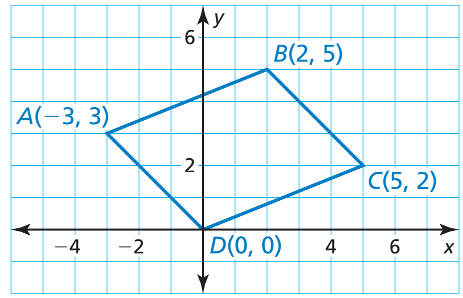
Ways to Prove a Quadrilateral Is a Parallelogram

1. Show that both pairs of opposite sides are parallel. *(Definition)*	
2. Show that both pairs of opposite sides are congruent. *(Parallelogram Opposite Sides Converse)*	
3. Show that both pairs of opposite angles are congruent. *(Parallelogram Opposite Angles Converse)*	
4. Show that one pair of opposite sides are congruent and parallel. *(Opposite Sides Parallel and Congruent Theorem)*	
5. Show that the diagonals bisect each other. *(Parallelogram Diagonals Converse)*	

Using Coordinate Geometry

EXAMPLE 5 **Identifying a Parallelogram in the Coordinate Plane**

Show that quadrilateral *ABCD* is a parallelogram.



SOLUTION

Method 1 Show that a pair of sides are congruent and parallel. Then apply the Opposite Sides Parallel and Congruent Theorem.

First, use the Distance Formula to show that $\overline{AB}$ and $\overline{CD}$ are congruent.

$$AB = \sqrt{[2-(-3)]^2 + (5-3)^2} = \sqrt{29}$$

$$CD = \sqrt{(5-0)^2 + (2-0)^2} = \sqrt{29}$$

Because $AB = CD = \sqrt{29}$, $\overline{AB} \cong \overline{CD}$.

Then, use the slope formula to show that $\overline{AB} \parallel \overline{CD}$.

$$\text{slope of } \overline{AB} = \frac{5-3}{2-(-3)} = \frac{2}{5}$$

$$\text{slope of } \overline{CD} = \frac{2-0}{5-0} = \frac{2}{5}$$

Because $\overline{AB}$ and $\overline{CD}$ have the same slope, they are parallel.

▶ $\overline{AB}$ and $\overline{CD}$ are congruent and parallel. So, *ABCD* is a parallelogram by the Opposite Sides Parallel and Congruent Theorem.

Method 2 Show that opposite sides are congruent. Then apply the Parallelogram Opposite Sides Converse. In Method 1, you already have shown that because $AB = CD = \sqrt{29}$, $\overline{AB} \cong \overline{CD}$. Now find *AD* and *BC*.

$$AD = \sqrt{(-3-0)^2 + (3-0)^2} = 3\sqrt{2}$$

$$BC = \sqrt{(2-5)^2 + (5-2)^2} = 3\sqrt{2}$$

Because $AD = BC = 3\sqrt{2}$, $\overline{AD} \cong \overline{BC}$.

▶ $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$. So, *ABCD* is a parallelogram by the Parallelogram Opposite Sides Converse.

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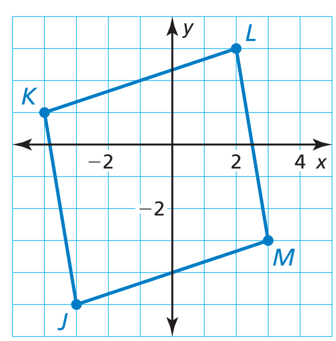
7. Show that quadrilateral *JKLM* is a parallelogram.

8. Refer to the Concept Summary on page 379. Explain two other methods you can use to show that quadrilateral *ABCD* in Example 5 is a parallelogram.



7.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. WRITING A quadrilateral has four congruent sides. Is the quadrilateral a parallelogram? Justify your answer.

2. DIFFERENT WORDS, SAME QUESTION Which is different? Find "both" answers.

Construct a quadrilateral with opposite sides congruent.	Construct a quadrilateral with one pair of parallel sides.
Construct a quadrilateral with opposite angles congruent.	Construct a quadrilateral with one pair of opposite sides congruent and parallel.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, state which theorem you can use to show that the quadrilateral is a parallelogram. *(See Examples 1 and 3.)*

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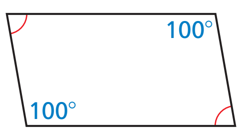

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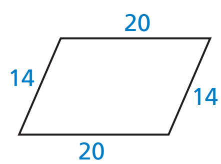

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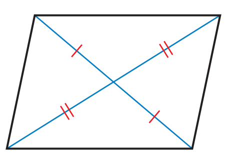

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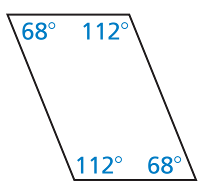

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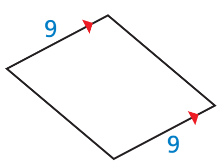

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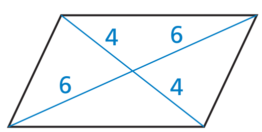
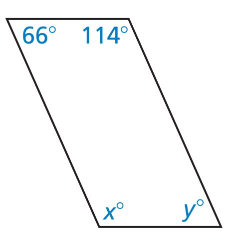


In Exercises 9–12, find the values of x and y that make the quadrilateral a parallelogram. *(See Example 2.)*

9.



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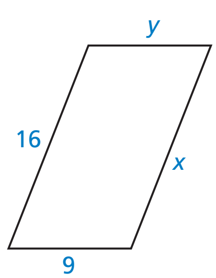

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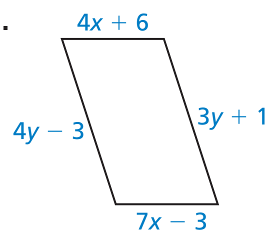

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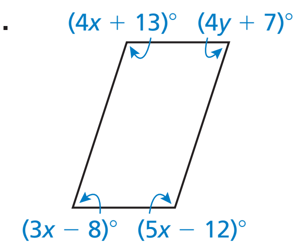
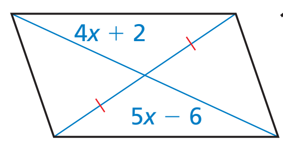


In Exercises 13–16, find the value of x that makes the quadrilateral a parallelogram. *(See Example 4.)*

13.



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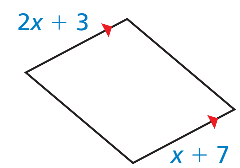

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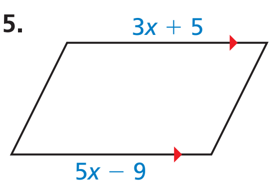

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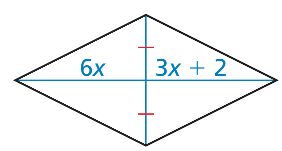


In Exercises 17–20, graph the quadrilateral with the given vertices in a coordinate plane. Then show that the quadrilateral is a parallelogram. *(See Example 5.)*

17. $A(0, 1)$, $B(4, 4)$, $C(12, 4)$, $D(8, 1)$

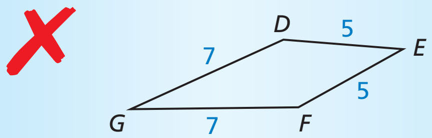
18. $E(-3, 0)$, $F(-3, 4)$, $G(3, -1)$, $H(3, -5)$

19. $J(-2, 3)$, $K(-5, 7)$, $L(3, 6)$, $M(6, 2)$

20. $N(-5, 0)$, $P(0, 4)$, $Q(3, 0)$, $R(-2, -4)$

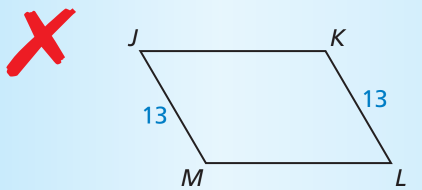
ERROR ANALYSIS **In Exercises 21 and 22, describe and correct the error in identifying a parallelogram.**

21.



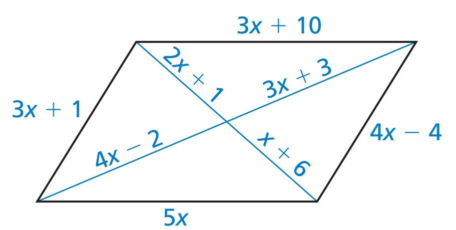
DEFG is a parallelogram by the Parallelogram Opposite Sides Converse (Theorem 7.7).

22.

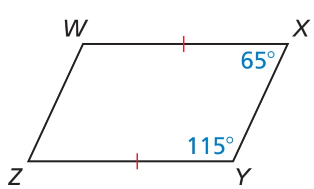


JKLM is a parallelogram by the Opposite Sides Parallel and Congruent Theorem (Theorem 7.9).

23. **MATHEMATICAL CONNECTIONS** What value of x makes the quadrilateral a parallelogram? Explain how you found your answer.



24. **MAKING AN ARGUMENT** Your friend says you can show that quadrilateral *WXYZ* is a parallelogram by using the Consecutive Interior Angles Converse (Theorem 3.8) and the Opposite Sides Parallel and Congruent Theorem (Theorem 7.9). Is your friend correct? Explain your reasoning.



ANALYZING RELATIONSHIPS **In Exercises 25–27, write the indicated theorems as a biconditional statement.**

25. Parallelogram Opposite Sides Theorem (Theorem 7.3) and Parallelogram Opposite Sides Converse (Theorem 7.7)

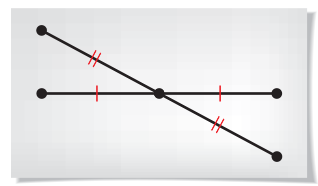
26. Parallelogram Opposite Angles Theorem (Theorem 7.4) and Parallelogram Opposite Angles Converse (Theorem 7.8)

27. Parallelogram Diagonals Theorem (Theorem 7.6) and Parallelogram Diagonals Converse (Theorem 7.10)

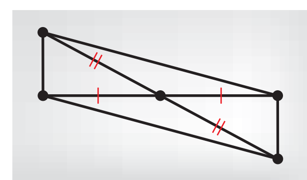
28. **CONSTRUCTION** Describe a method that uses the Opposite Sides Parallel and Congruent Theorem (Theorem 7.9) to construct a parallelogram. Then construct a parallelogram using your method.

29. **REASONING** Follow the steps below to construct a parallelogram. Explain why this method works. State a theorem to support your answer.

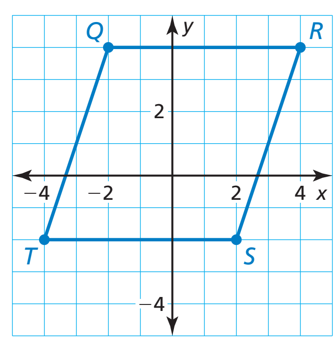
Step 1 Use a ruler to draw two segments that intersect at their midpoints.



Step 2 Connect the endpoints of the segments to form a parallelogram.

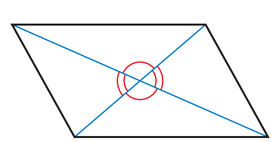


30. **MAKING AN ARGUMENT** Your brother says to show that quadrilateral *QRST* is a parallelogram, you must show that $\overline{QR} \parallel \overline{TS}$ and $\overline{QT} \parallel \overline{RS}$. Your sister says that you must show that $\overline{QR} \cong \overline{TS}$ and $\overline{QT} \cong \overline{RS}$. Who is correct? Explain your reasoning.

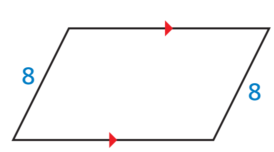


REASONING **In Exercises 31 and 32, your classmate incorrectly claims that the marked information can be used to show that the figure is a parallelogram. Draw a quadrilateral with the same marked properties that is clearly *not* a parallelogram.**

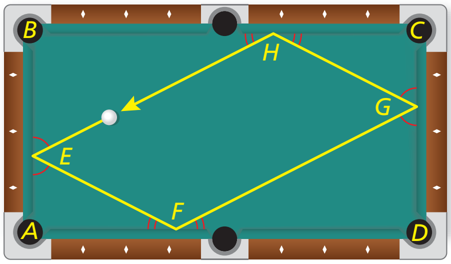
31.



32.



33. MODELING WITH MATHEMATICS You shoot a pool ball, and it rolls back to where it started, as shown in the diagram. The ball bounces off each wall at the same angle at which it hits the wall.



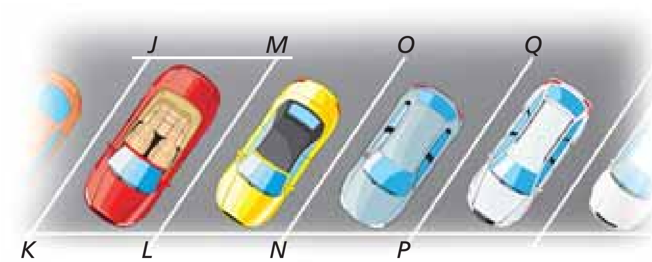
a. The ball hits the first wall at an angle of 63°. So $m\angle AEF = m\angle BEH = 63°$. What is $m\angle AFE$? Explain your reasoning.

b. Explain why $m\angle FGD = 63°$.

c. What is $m\angle GHC$? $m\angle EHB$?

d. Is quadrilateral *EFGH* a parallelogram? Explain your reasoning.

34. MODELING WITH MATHEMATICS In the diagram of the parking lot shown, $m\angle JKL = 60°$, $JK = LM = 21$ feet, and $KL = JM = 9$ feet.



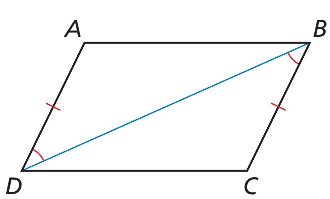
a. Explain how to show that parking space *JKLM* is a parallelogram.

b. Find $m\angle JML$, $m\angle KJM$, and $m\angle KLM$.

c. $\overline{LM} \parallel \overline{NO}$ and $\overline{NO} \parallel \overline{PQ}$. Which theorem could you use to show that $\overline{JK} \parallel \overline{PQ}$?

REASONING **In Exercises 35–37, describe how to prove that *ABCD* is a parallelogram.**

35.



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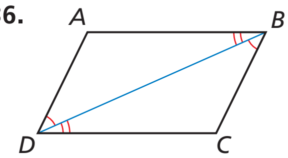
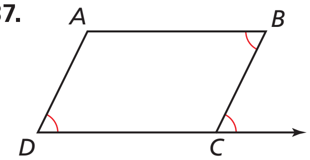
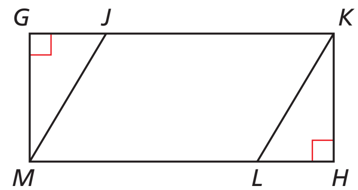


37.



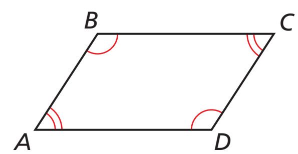
38. REASONING Quadrilateral *JKLM* is a parallelogram. Describe how to prove that $\triangle MGJ \cong \triangle KHL$.



39. PROVING A THEOREM Prove the Parallelogram Opposite Angles Converse (Theorem 7.8). (*Hint*: Let $x°$ represent $m\angle A$ and $m\angle C$. Let $y°$ represent $m\angle B$ and $m\angle D$. Write and simplify an equation involving x and y.)

Given $\angle A \cong \angle C$, $\angle B \cong \angle D$

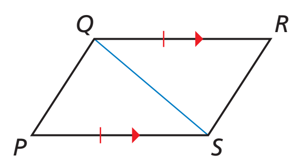
Prove *ABCD* is a parallelogram.



40. PROVING A THEOREM Use the diagram of *PQRS* with the auxiliary line segment drawn to prove the Opposite Sides Parallel and Congruent Theorem (Theorem 7.9).

Given $\overline{QR} \parallel \overline{PS}$, $\overline{QR} \cong \overline{PS}$

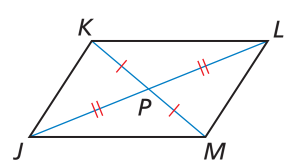
Prove *PQRS* is a parallelogram.



41. PROVING A THEOREM Prove the Parallelogram Diagonals Converse (Theorem 7.10).

Given Diagonals $\overline{JL}$ and $\overline{KM}$ bisect each other.

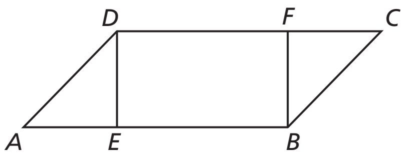
Prove *JKLM* is a parallelogram.



42. PROOF Write a proof.

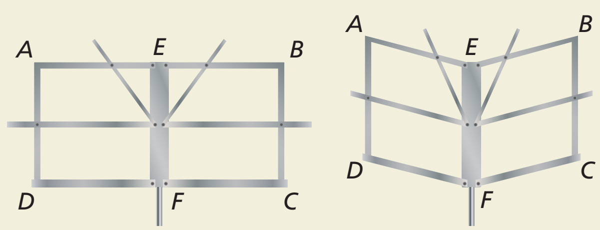
Given *DEBF* is a parallelogram.
$AE = CF$

Prove *ABCD* is a parallelogram.

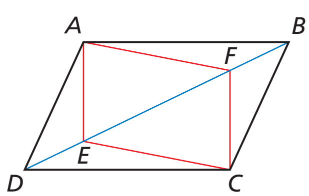


43. REASONING Three interior angle measures of a quadrilateral are 67°, 67°, and 113°. Is this enough information to conclude that the quadrilateral is a parallelogram? Explain your reasoning.

44. HOW DO YOU SEE IT? A music stand can be folded up, as shown. In the diagrams, *AEFD* and *EBCF* are parallelograms. Which labeled segments remain parallel as the stand is folded?



45. CRITICAL THINKING In the diagram, *ABCD* is a parallelogram, $BF = DE = 12$, and $CF = 8$. Find AE. Explain your reasoning.



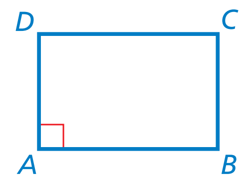
46. THOUGHT PROVOKING Create a regular hexagon using congruent parallelograms.

47. WRITING The Parallelogram Consecutive Angles Theorem (Theorem 7.5) says that if a quadrilateral is a parallelogram, then its consecutive angles are supplementary. Write the converse of this theorem. Then write a plan for proving the converse. Include a diagram.

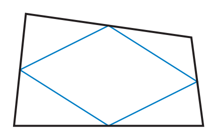
48. PROOF Write a proof.

Given *ABCD* is a parallelogram.
$\angle A$ is a right angle.

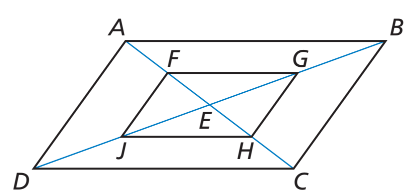
Prove $\angle B$, $\angle C$, and $\angle D$ are right angles.



49. ABSTRACT REASONING The midpoints of the sides of a quadrilateral have been joined to form what looks like a parallelogram. Show that a quadrilateral formed by connecting the midpoints of the sides of any quadrilateral is always a parallelogram. (*Hint*: Draw a diagram. Include a diagonal of the larger quadrilateral. Show how two sides of the smaller quadrilateral relate to the diagonal.)



50. CRITICAL THINKING Show that if *ABCD* is a parallelogram with its diagonals intersecting at *E*, then you can connect the midpoints *F*, *G*, *H*, and *J* of $\overline{AE}$, $\overline{BE}$, $\overline{CE}$, and $\overline{DE}$, respectively, to form another parallelogram, *FGHJ*.

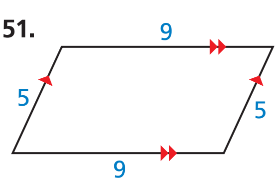


Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Classify the quadrilateral. *(Skills Review Handbook)*

51.



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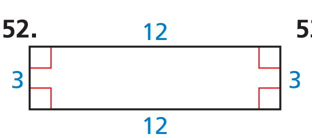

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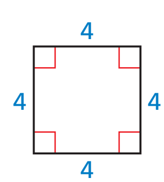

54.